



# Lorentzian, Affine, and Hyperbolic Differential Geometry

*In Memory of Todd Drumm*

## ABOUT THE WORKSHOP

This workshop is devoted to the mathematical legacy of Todd Drumm. The primary topics are affine manifolds, constant-curvature Lorentzian geometry, and links with hyperbolic geometry, as well as the geometry of the bidisk.

## ORGANIZERS

**Charles Frances**, University of Strasbourg  
**Bill Goldman**, University of Maryland

**Jill McGowan**, Howard University  
**Karin Melnick**, University of Luxembourg

## PARTICIPANTS

**Thierry Barbot**, Avignon Universite  
**Ara Basmajian**, City University of New York  
**Jean-Philippe Burelle**, University of Sherbrooke  
**Virginie Charette**, University of Sherbrooke  
**Charles Daly**, Brown University  
**Jeff Danciger**, University of Texas  
**Sourav Ghosh**, Ashoka University  
**Krishnendu Gongopadhyay**, ISER Mohali  
**François Guéritaud**, University of Strasbourg

**Aimee Johnson**, Swarthmore College  
**Alessandra Iozzi**, ETH Zurich  
**Youngju Kim**, Konkuk University  
**Sean Lawton**, George Mason University  
**Sara Maloni**, University of Virginia  
**John Parker**, Durham University  
**Jean-Marc Schlenker**, University of Luxembourg  
**Ilia Smilga**, University of Oxford  
**Ser-Peow Tan**, National University of Singapore  
**Neza Zager Korenjak**, University of Michigan



CSIC Building, 4th Floor  
8169 Paint Branch Drive  
University of Maryland  
College Park, MD 20742





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## Schedule at a Glance

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	Monday	Tuesday	Wednesday
8:00			
9:00	Breakfast	Breakfast	Breakfast
10:00	Barbot	Lawton	Schlenker
11:00	Burelle	Parker	Daly
12:00	Coffee Break	Coffee Break	Maloni
13:00	Lunch	Lunch	Lunch
14:00	Smilga	Basmajian	
15:00	Coffee Break	Coffee Break	
16:00	Zager Korenjak	Gongopadhyay	
17:00	Danciger	Tan	
18:00		Reception	

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# Workshop Overview

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This workshop is devoted to the mathematical legacy of Todd Drumm. The primary topics are affine manifolds, constant-curvature Lorentzian geometry, and links with hyperbolic geometry, as well as the geometry of the bidisk.

Since Margulis' discovery in 1983 of properly discontinuous, affine actions of nonabelian free groups on 3-dimensional Minkowski space, the corresponding Margulis space-times were the focus of robust research activity in affine and Lorentzian differential geometry. This activity includes Drumm's construction of fundamental domains bounded by crooked planes; the characterization of proper actions via the continuous Margulis invariant of Goldman, Labourie, and Margulis; and the tameness of Margulis space-times proved by Choi, Danciger, Guéritaud, and Kassel. From the same era that gave rise to Margulis space-times, famous conjectures in affine geometry such as the Auslander Conjecture have seen substantial but limited progress and remain very much open.

Closely related to affine geometry and hyperbolic geometry are constant-curvature Lorentzian structures, convex projective structures, and others. The workshop will bring together leading experts in these areas to build on past achievements and discuss current developments and future challenges. Todd Drumm received his Ph.D. from the University of Maryland in 1990. He was a professor at Howard University from 2008 until his sudden passing in March 2020.

## Organizing committee

CHARLES FRANCES, University of Strasbourg

WILLIAM GOLDMAN, University of Maryland

JILL MCGOWAN, Howard University

KARIN MELNICK, University of Luxemburg

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# Workshop Schedule

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## MONDAY, APRIL 7, 2025

- 8:30 - 9:00      BREAKFAST
- 9:00 - 9:10      DORON LEVY (University of Maryland/Director, Brin MRC)  
*Opening*
- 9:10 - 10:10     THIERRY BARBOT (Avignon University)  
*From Crooked Planes to Multi-Black Holes*
- 10:10 - 11:10    JEAN-PHILIPPE BURELLE (University of Sherbrooke)  
*Higher Crooked Planes*
- 11:10 - 11:40    COFFEE BREAK
- 11:40 - 12:40    FRANCOIS GUERITAUD (University of Strasbourg)  
*The Walls of the Space of Admissible Deformations*
- 12:40 - 2:00     LUNCH
- 2:00 - 3:00      ILIA SMILGA (University of Oxford)  
*Properness Criteria for Affine Actions of Anosov Groups*
- 3:00 - 3:30      COFFEE BREAK
- 3:30 - 4:30      NEZA ZAGER KORENJAK (University of Michigan)  
*Constructing Proper Affine Actions*
- 4:30 - 5:30      JEFF DANCIGER (University of Texas)  
*Thurston's Asymmetric Metric on Teichmüller Space Revisited*

## TUESDAY, APRIL 8, 2025

- 8:30 - 9:00      BREAKFAST
- 9:00 - 10:00    SEAN LAWTON (George Mason University)  
*The  $SU(2,1)$ -Character Variety of the 1-Holed Torus*
- 10:00 - 11:00    JOHN PARKER (Durham University)  
*Non-Arithmetic Complex Hyperbolic Lattices*
- 11:00 - 11:30    COFFEE BREAK
- 11:30 - 11:40    GROUP PHOTO
- 11:40 - 12:40    YOUNGJU KIM (Konkuk University)  
*Tubes in Complex Hyperbolic Manifolds*
- 12:40 - 2:00     LUNCH
- 2:00 - 3:00      ARA BASMAJIAN (The City University of New York)  
*A Bers Type Classification of Big Mapping Classes*
- 3:00 - 3:30      COFFEE BREAK
- 3:30 - 4:30      KRISHNENDU GONGOPADHYAY (ISER Mohali)  
*Reciprocity in Hecke Groups*
- 4:30 - 5:30      SER-PEOW TAN (National University of Singapore)  
*Punctured Torus Groups, Generalised Gauss Maps and Billiards on Hyperbolic Quadrilaterals*
- 5:30 - 7:00      RECEPTION: A TIME FOR REMINISCENCES
- 7:00 - 10:00     CONFERENCE DINNER

## WEDNESDAY, APRIL 9, 2025

- 8:30 - 9:00      BREAKFAST
- 9:00 - 10:00    JEAN-MARC SCHLENKER (University of Luxembourg)  
*Rigidity of Convex Domains with Prescribed Boundary Metric: Hyperbolic vs. Anti-de Sitter*
- 10:00 - 11:00    CHARLES DALY (Brown University)  
*Projective Rigidity of Once Punctured Torus Bundles and Twisted Alexander Polynomials*
- 11:00 - 11:15    COFFEE BREAK
- 11:15 - 12:15    SARA MALONI (University of Virginia)  
*Topology of the Space of  $D$ -pleated Surfaces*
- 12:15 - 12:30    WORKSHOP CLOSING
- 12:30 - 1:30     LUNCH



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# Abstracts of talks

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## From Crooked Planes to Multi-Black Holes

**THIERRY BARBOT**

*Avignon University*

Monday, April 7, 2025 @ 9:10 AM

In this talk, I will present the known examples of globally hyperbolic spatially compact conformally flat spacetimes whose holonomy group preserves a point in the Einstein Universe. It includes extensions of the so-called "Margulis spacetimes," and a generalization of the "grafting" procedure in the context of conformally flat Riemannian closed manifolds. I will try to justify that conjecturally, these examples are the only possible ones.

## Higher Crooked Planes

**JEAN-PHILIPPE BURELLE**

*University of Sherbrooke*

Monday, April 7, 2025 @ 10:10 AM

Crooked Planes, introduced by Drumm in his 1990 thesis, are piecewise-linear surfaces which can be used to bound fundamental domains for proper affine actions of free groups on  $R^3$ . I will discuss how, surprisingly, similar hypersurfaces can be defined in higher dimensions and bound fundamental domains for the action of positive representations in  $SL(n, R)$  on projective spaces, as well as certain proper affine deformations of positive representations in  $SO(2n, 2n-1)$ .

# The Walls of the Space of Admissible Deformations

FRANCOIS GUERITAUD

*University of Strasbourg*

Monday, April 7, 2025 @ 11:40 AM

The deformation space of proper affine actions on Minkowski 3-space, with given linear part, is an open convex cone (and its negative) corresponding to infinitesimal deformations of the associated hyperbolic surface that lengthen all the curves. The supporting hyperplanes of this "admissible" cone should correspond to data in the surface, such as points of its curve complex. I will describe a version of this theory when the surface is decorated with ideal points and horoballs, which gives rise to interesting finite combinatorics (associahedra and some of their strange cousins) even when the group is trivial. In particular, each face of the admissible cone comes with a natural finite simplicial decomposition, which can be seen as an appropriate arc complex. I will also discuss some limiting phenomena when the group is infinite. This is a report on work of and joint work with Pallavi Panda.

## Properness Criteria for Affine Actions of Anosov Groups

ILIA SMILGA

*University of Oxford*

Monday, April 7, 2025 @ 2:00 PM

I will present some criteria (necessary or sufficient) for the action on the affine space of a group  $\Gamma$  of affine transformations to be proper. This is joint work with Fanny Kassel. The main of these criteria links properness of action to the divergence of a parameter called the Margulis invariant. This invariant measures roughly the translation part of an affine transformation, but in a way that is invariant by conjugation. This link was already known in some special cases (and has often been exploited to construct proper actions). We tried to establish it in as general setting as possible. We proved it in particular if  $\Gamma$  has some suitable Anosov property (with respect to some natural parabolic subgroup, that depends on the affine group we are working in). I will possibly also evoke some other invariants similar to the Margulis invariant, that could lead to criteria that work in even more general settings.

# Constructing Proper Affine Actions

NEZA ZAGER KORENJAK

*University of Michigan*

Monday, April 7, 2025 @ 3:30 PM

Danciger-Gueritaud-Kassel used strip deformations to fully understand proper affine actions of free groups in three dimensions. We generalize some of their ideas to the setting of proper affine actions of Fuchsian and positive representations of free groups acting on  $R^4n - 1$  by defining versions of higher strip deformations. We can do this using the Margulis invariant or by constructing Drumm-like fundamental domains. Part of this work is joint with Jean-Philippe Burelle.

# Thurston's Asymmetric Metric on Teichmüller Space Revisited

JEFF DANCIGER

*University of Texas*

Monday, April 7, 2025 @ 4:30 PM

A surface  $S$  of negative Euler characteristic admits many different hyperbolic structures. These are organized by the classical Teichmüller space of  $S$ . Each hyperbolic structure determines (and is determined by) a length spectrum function, which associates to each homotopy class of closed curves on  $S$ , the length of the geodesic representative. One way to tell a pair of hyperbolic structures apart is to compare their length spectra; this idea leads to Thurston's asymmetric metric on Teichmüller space. We will recall some key points of Thurston's story and then generalize to find interesting geometrical/dynamical invariants associated to a  $d$ -tuple of hyperbolic structures ( $d > 2$ ) and, most generally, to a positive representation of  $\pi_1 S$  into a real split, semi-simple Lie group of higher rank. Joint work with Francois Gueritaud and Fanny Kassel.

# The $SU(2,1)$ -Character Variety of the 1-Holed Torus

SEAN LAWTON

*George Mason University*

Tuesday, April 8, 2025 @ 9:00 AM

We sketch the proof that the  $SU(2,1)$ -character variety of the 1-holed torus is homotopic to a product of circles. We then discuss the mapping class group dynamics on this character variety. In particular, we describe an open domain of discontinuity. This work represents collaborative work with Sara Maloni and Frederic Palesi. See arXiv:2402.10838 for more information (recently accepted for publication in *Moduli*).

# Non-Arithmetic Complex Hyperbolic Lattices

JOHN PARKER

*Durham University*

Tuesday, April 8, 2025 @ 10:00 AM

In a pioneering paper of 1980 Mostow constructed the first non-arithmetic lattices in  $PU(2, 1)$ , the holomorphic isometry group of complex hyperbolic 2-space. Subsequently, using different methods, Livne in 1981 and Deligne and Mostow in 1986 constructed further examples. This remained the state of the art until 2016 when more examples were constructed in my joint work with Deraux and Paupert. To do so, we revisited Mostow's original method of building fundamental domains. We refined this method in a paper of 2021 giving an algorithmic method of constructing fundamental domains which includes all previously known examples and also several more. In this talk, I will give an overview of this topic including discussion of different methods of construction used by different authors.

# Tubes in Complex Hyperbolic Manifolds

YOUNGJU KIM

*Konkuk University*

Tuesday, April 8, 2025 @ 11:40 AM

We will talk about a tubular neighborhood theorem for an embedded complex geodesic in a complex hyperbolic 2-manifold where the width of the tube depends only on the Euler characteristic of the embedded complex geodesic. We give an explicit estimate for this width. We supply two applications of the tubular neighborhood theorem, the first is a lower volume bound for such manifolds. The second is an upper bound on the first eigenvalue of the Laplacian in terms of the geometry of the manifold. This is a joint work with Ara Basmajian.

# A Bers Type Classification of Big Mapping Classes

ARA BASMAJIAN

*The City University of New York*

Tuesday, April 8, 2025 @ 2:00 PM

The first part of this talk will be an introduction to the geometry and topology of big surfaces (surfaces with an infinitely generated fundamental group). Big surfaces include the Cantor tree, Loch Ness monster, and flute surfaces. Such surfaces admit a rich variety of hyperbolic geometric structures which are studied in a number of different ways. For surfaces with a finitely generated fundamental group (for example, compact surfaces) any two geometric structures on it are related by a quasiconformal (qc) homeomorphism. Such homeomorphisms control how distorted the geometry of the surface can become. As a result, the space of hyperbolic metrics on a compact surface (of genus bigger than 1) called the Teichmüller space is given by quasiconformal deformations. On the other hand, for a big surface, there is no natural choice of Teichmüller space as there are uncountably many.

The second part of this talk will be on big mapping class groups. The mapping class group of a surface  $\Sigma$  is its group of self-homeomorphisms modulo isotopy. If the fundamental group of  $\Sigma$  is finitely generated, the mapping class group is finitely generated and acts faithfully on the Teichmüller space. On the other hand, for a big surface the mapping class group is infinitely generated (hence the name, big mapping class group) and does not act naturally on any Teichmüller space.

The final part of this talk is on our classification of big mapping classes and a construction of various spaces of hyperbolic structures for which the big mapping class group acts faithfully. As an application of our work, we show that a big mapping class group is not algebraically isomorphic to the modular group (quasiconformal mapping class group) of any hyperbolic surface. This is joint work with Yassin Chandran.

# Reciprocity in Hecke Groups

KRISHNENDU GONGOPADHYAY

*ISER Mohali*

Tuesday, April 8, 2025 @ 3:30 PM

A group element is called reciprocal when it is conjugate to its own inverse. We shall classify the reciprocal elements and parametrize the reciprocal classes in the Hecke groups. We shall estimate the growth rates for the set of reciprocal conjugacy classes in the Hecke groups. This talk is based on my joint work with Debattam Das.

# Punctured Torus Groups, Generalised Gauss Maps and Billiards on Hyperbolic Quadrilaterals

SER-PEOW TAN

*National University of Singapore*

Tuesday, April 8, 2025 @ 4:30 PM

We consider certain families of objects (groups, surfaces, billiard tables) parametrization by rationals in  $(0,1)$  from different viewpoints: geometric, group theoretic and dynamical and explore the orbit decomposition, under certain natural actions, of the rationals for different parameters in the parameter space. In particular, we show how various questions like the pseudo-modularity of punctured torus groups, orbit classes of generalized Gauss maps on the unit interval and billiards on ideal hyperbolic quadrilaterals are related.

# Rigidity of Convex Domains with Prescribed Boundary Metric: Hyperbolic vs. Anti-de Sitter

JEAN-MARC SCHLENKER

*University of Luxembourg*

Wednesday, April 9, 2025 @ 9:00 AM

Let  $M$  be a 3-manifold with boundary, which admits a convex co-compact hyperbolic metric. We consider an analogue in this setting of the classical Weyl problem: given a hyperbolic metric  $h$  on  $\partial M$  with curvature  $K \geq -1$ , is there a unique hyperbolic metric on  $M$  for which the boundary is convex, with induced metric  $h$ ? A "dual" question concerns the third fundamental form on the boundary, or, for a "minimal" convex subset, the measured bending lamination. We will describe several recent results due to several authors, some remaining open questions for hyperbolic manifolds, and the corresponding questions in the AdS setting (some going back to Mess). We will focus on statements which are still open for AdS spacetimes, and on a statement recently proved for hyperbolic manifold, whose AdS analog is false.

# Projective Rigidity of Once Punctured Torus Bundles and Twisted Alexander Polynomials

CHARLES DALY

*Brown University*

Wednesday, April 9, 2025 @ 10:00 AM

Closed hyperbolic 3-manifolds have long been known to be rigid geometric objects due to Mostow's celebrated rigidity theorem. If one considers the natural projective structure associated with a closed hyperbolic 3-manifold, one can ask if it admits any deformations of its hyperbolic structure in the larger group of projective transformations. In this talk, we address this question and a closely related notion, infinitesimal projective rigidity rel boundary, in the context of hyperbolic once-punctured torus bundles. We explain how the twisted Alexander polynomial associated to the holonomy representation can be used to detect this condition.



# Topology of the Space of D-pleated Surfaces

SARA MALONI

*University of Virginia*

Wednesday, April 9, 2025 @ 11:15 AM

In this talk, we define the notion of d-pleated surfaces, which is a higher rank generalization of the classical notion of (abstract) pleated surfaces in three-dimensional hyperbolic space. We give a description of the global topology of the space of d-pleated surfaces and prove that every connected component of the character variety contains exactly one connected component of the space of d-pleated surfaces. This is joint work with Giuseppe Martone, Filippo Mazzoli and Tengren Zhang.

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# The Brin Mathematics Research Center

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The Brin Mathematics Research Center is a research center that sponsors activity in all areas of pure and applied mathematics and statistics. The Brin MRC was funded in 2022 through a generous gift from the Brin Family. The Brin MRC is part of the Department of Mathematics at the University of Maryland, College Park.

Activities sponsored by the Brin MRC include long programs, conferences and workshops, special lecture series, and summer schools. The Brin MRC provides ample opportunities for short-term and long-term visitors that are interested in interacting with the faculty at the University of Maryland and in experiencing the metropolitan Washington DC area.

The mission of the Brin MRC is to promote excellence in mathematical sciences. The Brin MRC is home to educational and research activities in all areas of mathematics. The Brin MRC provides opportunities to the global mathematical community to interact with researchers at the University of Maryland. The center allows the University of Maryland to expand and showcase its mathematics and statistics research excellence nationally and internationally.

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# List of Participants

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TOBY ALDAPE, University of Texas  
THIERRY BARBOT, Avignon University  
ARA BASMAJIAN, The City University of New York  
HARRISON BRAY, George Mason University  
JEAN-PHILIPPE BURELLE, University of Sherbrooke  
ANUNOY CHAKRABORTY, George Mason University  
VIRGINIE CHARETTE, University of Sherbrooke  
CHARLES DALY, Brown University  
JEFF DANCIGER, University of Texas  
CHARLES FRANCES, University of Strasbourg  
BILL GOLDMAN, University of Maryland  
KRISHNENDU GONGOPADHYAY, ISER Mohali  
FRANCOIS GUERITAUD, University of Strasbourg  
ALESSANDRA IOZZI, ETH Zurich  
AIMEE JOHNSON, Swarthmore College  
YOUNGJU KIM, Konkuk University  
SEAN LAWTON, George Mason University  
DORON LEVY, University of Maryland/Director, Brin MRC  
GABRIEL LUMPKIN, George Mason University  
SARA MALONI, University of Virginia  
JILL MCGOWAN, Howard University  
KARIN MELNICK, University of Luxembourg  
JOHN PARKER, Durham University  
ALAN REID, Rice University  
PIER-OLIVIER RODRIGUE, George Mason University  
INYOUNG RYU, Texas A&M University  
JEAN-MARC SCHLENKER, University of Luxembourg  
ILIA SMILGA, University of Oxford  
SER-PEOW TAN, National University of Singapore  
NEZA ZAGER KORENJAK, University of Michigan